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observe: $|\vec{0}| = |0\vec{u}| = |0||\vec{u}|$

$|\vec{u}| = 0$ if and only if $\vec{u} = \vec{0}$

Definition: The direction of a vector $\vec{u}$ is the associated unit vector (i.e. vector w/ length 1):

i.e. Direction of $\vec{u}$ is $\frac{1}{|\vec{u}|}\vec{u}$ (unit vector) when $\vec{u} \neq \vec{0}$

Claim: $\frac{1}{|\vec{u}|}\vec{u}$ is a unit vector

$$\left|\frac{1}{|\vec{u}|}\vec{u}\right| = \left|\frac{1}{|\vec{u}|}\right| |\vec{u}| = \frac{1}{|\vec{u}|}|\vec{u}| = 1 \quad (\text{when } \vec{u} \neq \vec{0})$$

In $\mathbb{R}^3$, there are 3 special vectors called the component vectors:

$\vec{i} = \langle 1,0,0 \rangle$
$\vec{j} = \langle 0,1,0 \rangle$
$\vec{k} = \langle 0,0,1 \rangle$

"standard basis for $\mathbb{R}^3$"

Every vector is a sum of scalar multiples of component vectors

$\vec{u} = \langle u_1, u_2, u_3 \rangle$
$= \langle u_1, 0, 0 \rangle + \langle 0, u_2, 0 \rangle + \langle 0, 0, u_3 \rangle$
$= u_1\langle 1,0,0 \rangle + u_2\langle 0,1,0 \rangle + u_3\langle 0,0,1 \rangle$

8/30/21 $\boxed{12.3}$ Dot Product (goal: connect algebra of vectors to geometry via a new operation on vectors)

Def: Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3$

dot product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

(vector · vector → scalar)

$\vec{u}\cdot(\vec{v}\cdot\vec{w})$ doesn't make sense
- can't have vector · scalar
$[(\vec{u}\cdot\vec{v})]\vec{w}$ makes sense

Ex: $\vec{u} = \langle 1,3,5 \rangle$, $\vec{v} = \langle -3,5,7 \rangle$

$\vec{u}\cdot\vec{v} = (1)(-3) + (3)(5) + (5)(7) = -3 + 15 + 35 = 47$

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Theorem: (Properties of Dot Product)

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $\overset{\text{scalar}}{c} \in \mathbb{R}$

① $\vec{v}\cdot\vec{v} = v_1v_1 + v_2v_2 + v_3v_3 = v_1^2 + v_2^2 + v_3^2$
$= (\sqrt{v_1^2+v_2^2+v_3^2})^2 = |\vec{v}|^2$

② $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \vec{v}\cdot\vec{u}$
Dot product is commutative (not associative)

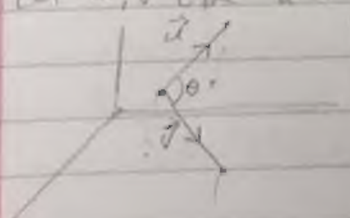
③ $\vec{u}\cdot(\vec{v}+\vec{w}) = \langle u_1, u_2, u_3\rangle \cdot \langle v_1+w_1, v_2+w_2, v_3+w_3\rangle$
$= u_1(v_1+w_1) + u_2(v_2+w_2) + u_3(v_3+w_3)$
$= (u_1v_1 + u_1w_1) + (u_2v_2 + u_2w_2) + (u_3v_3 + u_3w_3)$
$= (u_1v_1 + u_2v_2 + u_3v_3) + (u_1w_1 + u_2w_2 + u_3w_3)$
$= \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$
Dot product distributes over addition

④ $\vec{u}\cdot(c\vec{v}) = \langle u_1, u_2, u_3\rangle \cdot \langle cv_1, cv_2, cv_3\rangle$
$= u_1(cv_1) + u_2(cv_2) + u_3(cv_3) = c(u_1v_1 + u_2v_2 + u_3v_3)$
$= c(\vec{u}\cdot\vec{v})$

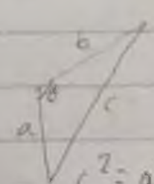
⑤ $\vec{0}\cdot\vec{v} = 0$

Theorem: (geometric interpretation of the dot product)

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and let θ be the angle b/t them. Then

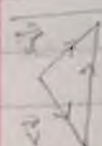


$\underset{\text{algebraic}}{\vec{u}\cdot\vec{v}} = \underset{\text{geometric}}{|\vec{u}||\vec{v}|\cos\theta}$



(generalization of Pythagorean theorem)

Proof: (recall law of cosines $c^2 = a^2 + b^2 - 2ab\cos(\theta)$)



consider $\vec{u}-\vec{v}$. Applying the law of cosines: $|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$

on the side we apply properties of Dot Product.

$|\vec{u}-\vec{v}|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = (\vec{u}-\vec{v})\cdot\vec{u} - (\vec{u}-\vec{v})\cdot\vec{v}$ →

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distribution of dot ... algebra

$= (\vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{u}) - (\vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{v}) = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v}$

$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - 2\vec{u}\cdot\vec{v}$ (comm. of dot)

$= |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u}\cdot\vec{v}$

$|\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$

$= |\vec{u} - \vec{v}|^2$

$= |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u}\cdot\vec{v}$

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

corollary: Supposing $\vec{u}$ and $\vec{v}$ are both non-zero

$\theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$

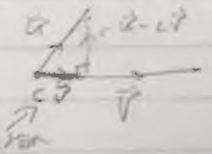
Observation: the zero-vector has an undefined angle with all other vectors

Corolary: If $\vec{u}$ and $\vec{v}$ are perpendicular (i.e orthogonal), then
$\vec{u}\cdot\vec{v} = 0$

conversly - $\vec{u}\cdot\vec{v} = 0$ implies $\vec{u}$ and $\vec{v}$ are orthogonal

Orthogonal Projection

Suppose $\vec{u}, \vec{v} \in \mathbb{R}^n$



To project $\vec{u}$ orthogonally onto $\vec{v}$:

$c\vec{v}\cdot(\vec{u} - c\vec{v}) = 0$

iff $c(\vec{v}\cdot\vec{u}) - c^2(\vec{v}\cdot\vec{v}) = 0$

$c(\vec{v}\cdot\vec{u} - c|\vec{v}|^2) = 0$ if either $c = 0$ or $\vec{u}\cdot\vec{v} - c|\vec{v}|^2 = 0$

So Assuming $|\vec{v}| \neq 0$ and $c \neq 0$, $c = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}$

projection of $\vec{u}$ onto $\vec{v}$

Def: The orthogonal projection of $\vec{u}$ onto $\vec{v}$ is

$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$

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$$\mathrm{Proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\Big(\underbrace{\frac{1}{|\vec{v}|}\vec{v}}_{\text{unit vector}}\Big)$$

$$= \mathrm{comp}_{\vec{v}}(\vec{u})\left(\frac{1}{|\vec{v}|}\vec{v}\right)$$

is the scalar projection of $\vec{u}$ onto $\vec{v}$ is $\mathrm{comp}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$

<u>Direction angles</u>: (angles w/ $\vec{i}$, $\vec{j}$ and $\vec{k}$)

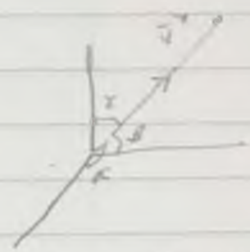
Let $\vec{v} \in \mathbb{R}^3$

the direction angles of $\vec{v}$ are the angles $\vec{v}$ makes $\vec{i}$ $\vec{j}$ and $\vec{k}$

i.e. $\alpha = \arccos\left(\frac{\vec{v}\cdot\vec{i}}{|\vec{v}||\vec{i}|}\right) = \arccos\left(\frac{v_1}{|\vec{v}|}\right)$

$\beta = \arccos\left(\frac{v_2}{|\vec{v}|}\right)$

$\gamma = \arccos\left(\frac{v_3}{|\vec{v}|}\right)$



<u>Note</u>: the direction angles determine the "would be location" of $\vec{v}$ on the unit sphere about the origin.

<u>Exercise</u>: show that any two of the direction angles of $\vec{v}$ determine the third...